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Locality, Quantum Many-Body Dynamics, and Gapped Ground State Phases.

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Outline

- I. Locality in quantum lattice systems
- II. Lieb-Robinson bounds and infinite system dynamics
- III. The quasi-adiabatic evolution
- IV. Gapped ground state phases
- V. Stability of spectral gaps
- VI. Invariants of gapped phases

IV. Gapped ground state phases

Ground states:

$$\|\Phi\|_F = \sup_{x,y\in\Gamma} \frac{1}{F(d(x,y))} \sum_{\substack{X \\ x,y\in X}} \|\Phi(X)\|$$

Consider the class of systems defined by interactions Φ with $\|\Phi\|_F < \infty$, for some $F-$function F, for example $F(r) = F_0(r)e^{-ar^\theta}$, $a > 0, \theta \in (0,1]$ $(\Phi \in \mathcal{B}_{a,\theta})$.

Then, we can define a derivation $\delta : \mathcal{A}_{\rm loc} \to \mathcal{A}_\Gamma$, by

unbounded

$$\delta^\Phi(A) = \sum_{X, X\cap Y\neq\emptyset} [\Phi(X), A], \quad A \in \mathcal{A}_Y, \text{ finite } Y \subset \Gamma,$$

$$\tau_t^\Phi(A) = e^{it\delta^\Phi}(A)$$

which has a closure that is the generator of the dynamics τ_t^Φ, again denoted by δ^Φ.

A state ω on $\mathcal{A}$ is a ground state for the dynamics τ_t with generator δ if

$$\boxed{\omega(A^*\delta(A)) \geq 0, \text{ for all } A \in \operatorname{dom}\delta.}$$

$\omega : \mathcal{A} \to \mathbb{C}$
linear, $\omega(1) = 1$
$\omega(A^*A) \geq 0$

It is sufficient to check the ground state inequality for A in a core for δ, such as $\mathcal{A}_{\rm loc}$.

$$\cong e^{it[H,\cdot]} = e^{itH} \cdot e^{-itH}$$

$$e^{it\delta} = id + it\delta + \frac{(it)^2}{2!}\delta^2 + \cdots$$

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X) \quad ; \quad \Lambda_n \nearrow \Gamma$$

$$H_{\Lambda_n} \Psi_{\Lambda_n} = \lambda^0_{\Lambda_n} \Psi_{\Lambda_n} \qquad \lambda^0_{\Lambda_n} = \text{ground state energy.}$$

$$\omega_{\Lambda_n}(A) = \langle \Psi_{\Lambda_n}, A \Psi_{\Lambda_n} \rangle$$

$$\downarrow$$

$$\omega(A) : \quad \omega(A^* \delta^{\Phi}(A)) \geq 0$$

For finite systems this definition identifies the states of minimal energy (expectation of H_Λ).

Infinite volume limits of finite-volume ground states are ground states.

Gapped ground states

Assume δ^Φ has a finite number of mutually disjoint pure ground states: $\mathcal{S}^\Phi = \{\omega_1, \ldots, \omega_n\}$.

We say that the ground states are gapped if there exists $\gamma > 0$, such that for all $i = 1, \ldots, n$, we have

$$\omega_i(A^* \delta^\Phi(A)) \geq \gamma \omega_i(A^* A), \text{ for all } A \in \mathcal{A}^{\mathrm{loc}}, \text{ with } \omega_i(A) = 0.$$

This is equivalent to saying that the GNS Hamiltonian for the system in the ground state ω_i is a non-negative self-adjoint operator with a one-dimensional kernel and a spectral gap above zero of size $\geq \gamma$.

for any state ω on $\mathcal{Q}_\Gamma$, there exists a Hilbert space $\mathcal{H}_\omega$, a representation $\pi_\omega : \mathcal{Q}_\Gamma \to \mathcal{B}(\mathcal{H}_\omega)$, and vector $\Omega_\omega \in \mathcal{H}_\omega$ s.t. $\omega(A) = \langle \Omega_\omega, \pi_\omega(A) \Omega_\omega \rangle$

This rep is called the GNS rep of the state ω

!! it is unique up to unitary equivalence

meaning if $\mathcal{H}_1$, $\mathcal{H}_2$, π_1, π_2, Ω_1, Ω_2

$\pi_i(\mathfrak{A})\Omega_i$ is dense in $\mathcal{H}_i$

then $\exists$ $U : \mathcal{H}_1 \to \mathcal{H}_2$ s.t.

$$U \pi_1(A) U^* = \pi_2(A)$$

$$U \Omega_1 = \Omega_2$$

$$\pi_1 \underset{u}{\sim} \pi_2$$

If ω_1 and ω_2 have unitarily equiv. GNS reps we say $\omega_1 \underset{u}{\sim} \omega_2$.

if ω is a g.s. for Φ

$$\omega \circ \tau_t^{\Phi} = \omega \quad \forall t \in \mathbb{R}$$

$(\mathcal{H}_\omega, \underbrace{\pi_\omega \circ \tau_t^{\Phi}}, \Omega_\omega)$ are also GNS rep of ω.

$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$

$\omega(\tau_t^{\Phi}(A)) = \langle \Omega, \boxed{\pi \circ \tau_t^{\Phi}}(A)\Omega \rangle$

$\Longrightarrow \exists\ U_t$ unitary on $\mathcal{H}_\omega$

$$\pi_\omega \circ \tau_t^{\Phi} = U_t^* \pi_\omega(\cdot) U_t$$

$t \mapsto U_t$ strongly continuous.

Stone's Thm $\Rightarrow$ $U_t = e^{-itH_\omega}$

if ω is g.s.

H_ω : s.a. $H_\omega \geq 0$, $H_\omega \Omega_\omega = 0$

if gap $\geq \gamma > 0$, $\mathrm{spec}(H_\omega) \subseteq \{0\} \cup [\gamma, \infty)$

Examples:

- AKLT chain: $H = \sum_x P^{(2)}_{x,x+1}$

  $\Gamma^H = \mathbb{Z}$

  spin 1 chain ($n_x = 3$)

  * has unique g.s.
  * and it is gapped

- spin 1 chain $H = \sum_x P^{(2)}_{x,x+1} + P^{(1)}_{x,x+1}$

  $$= \sum_x \left(\mathbb{1} - P^{(0)}_{x,x+1}\right)$$

  - has 2 ground states
    are 2-periodic
  - both are gapped.

Gapped Ground State Phases

Consider, for a fixed choice of Γ and $n_x, x \in \Gamma$, the set $\mathcal{B}_{a,\theta}^{\text{gapped}}$ of all interactions $\Phi \in \mathcal{B}_{a,\theta}$, some $a > 0, \theta \in (0, 1]$, such that δ^Φ has a finite set of gapped ground states. Further restrictions can be imposed (uniqueness, symmetries ...).

Then, a gapped ground state phase is an equivalence class for an equivalence relation defined on $\mathcal{B}_{a,\theta}^{\text{gapped}}$.

Mathematical definitions

Suppose Φ_0 and Φ_1 are two interactions in the class $\mathcal{B}_{a,\theta}$, with ground state sets $\mathcal{S}^{\Phi_0}$ and $\mathcal{S}^{\Phi_1}$, respectively.

Definition 1. (Equivalence of interactions)

The interactions Φ_0 and Φ_1 belong to the same phase if there exists a differentiable curve of interactions $[0,1] \ni s \mapsto \Phi(s)$ in $\mathcal{B}_{a,\theta}$ such that

1. $\Phi(0) = \Phi_0, \Phi(1) = \Phi_1$;
2. There exists a constant $\gamma' > 0$, such that for all $s \in [0,1]$ $\Phi(s)$ has gapped ground states with gap $\gamma' > 0$.
3. There exist $a' > 0, \theta' \in (0,1]$, such that $\Phi(\cdot) \in \mathcal{B}^1_{a',\theta'}([0,1])$, defined as the Banach space of interactions for which, with $F(r) = e^{-a' r^{\theta'}} F_0(r)$,

$$\sup_{x,y\in\Gamma} \frac{1}{F(d(x,y))} \sum_{\text{finite} X: x,y\in X} \|\Phi(X,s)\| + |X| \|\Phi'(X,s)\|$$

   is a bounded by a bounded measurable function of s.

$\mathcal{S}_0 = \{\omega_0^1, \dots, \omega_0^n\}$; $\mathcal{S}_1 = \{\omega_1^1, \dots, \omega_1^n\}$

Suppose $\mathcal{S}_0$ and $\mathcal{S}_1$ are two finite sets of pure states of $\mathcal{A}_\Gamma$.

Definition 2. (Equivalence of states)

The sets of states $\mathcal{S}_0$ and $\mathcal{S}_1$ are automorphically equivalent (in the stretched exponential locality class) if there exists a continuous curve of interactions $[0,1] \ni s \mapsto \Psi(s)$ such that

1. There exist $a' > 0, \theta' \in (0,1]$, such that for all $s \in [0,1]$, $\Psi(s) \in \mathcal{B}_{a',\theta'}$;
2. $[0,1] \ni s \mapsto \Psi(s)$ is piecewise continuous in the norm of $\mathcal{B}_{a',\theta'}([0,1])$;
3. The family of automorphisms $\alpha_{s,0}$ generated by $\Psi(s)$ satisfies

$$\mathcal{S}_1 = \{\omega_i^0 \circ \alpha_{1,0} \mid \omega_i^0 \in \mathcal{S}_0\}.$$

It is easy to show that these two definitions define equivalence relations.

We would like to define a gapped ground state phase as an equivalence class. Which definition should we use?

Theorem (Equivalence of the Defs 1 and 2, N arXiv:2205.10460)

(i) (Def 2 $\implies$ Def 1) Let $\mathcal{S}_0$ be a set of mutually disjoint pure ground states gap bounded below by $\gamma > 0$ for the dynamics with generator δ_0 defined by an interaction $\Phi_0 \in \mathcal{B}_{a,\theta}$, for some $a > 0, \theta \in (0,1]$. If a set of states $\mathcal{S}_1$ is automorphically equivalent to $\mathcal{S}_0$ in the stretched exponential locality class, then there exists a differentiable curve of interactions of class $\mathcal{B}^1_{a',\theta'}([0,1])$, for some $a' > 0, \theta' \in (0,1]$, $\Phi(s), s \in [0,1]$, with $\Phi(0) = \Phi_0$, and such that $\mathcal{S}_1$ are gapped ground states with gap bounded below by γ for the dynamics generated by $\Phi(1)$.

(ii) (Def 1 $\implies$ Def 2) Suppose $s \mapsto \Phi(s)$ is a differentiable curve of interactions of class $\mathcal{B}^1_{a,\theta}([0,1])$, such that there exists $\gamma > 0$ and sets of mutually disjoint pure gapped ground states $\mathcal{S}_s$, $s \in [0,1]$, with gap bounded below by γ. Then, there exists as strongly continuous curve of automorphisms α_s of class $\mathcal{B}_{a',\theta'}([0,1])$, such that

$$\mathcal{S}_s = \{\omega \circ \alpha_{s,0} \mid \omega \in \mathcal{S}_0\}.$$

This a mathematical version of the definition of 'gapped phase' given in Xie Chen, Zheng-Cheng Gu, Xiao-Gang Wen, Phys. Rev. B 82, 155138 (2010).

V. Stability of the ground state gap Recap: Ground states

A state ω on $\mathcal{A}$ is a ground state for the dynamics τ_t with generator δ if

$$\omega(A^*\delta(A)) \geq 0, \text{ for all } A \in \mathrm{dom}\,\delta.$$

It is sufficient to check this condition for A in a core for δ, such as $\mathcal{A}_{\mathrm{loc}}$.

The GNS representation

The GNS representation of a state on $\mathcal{A}_\Gamma$ is given by a Hilbert space $\mathcal{H}$, a representation π of $\mathcal{A}$ on $\mathcal{H}$, and a cyclic vector $\Omega \in \mathcal{H}$ such that, for all $A \in \mathcal{A}$

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle, \quad A \in \mathcal{A}_\Gamma.$$

For ground states one finds that τ_t is implemented by a strongly continuous group of unitaries on $\mathcal{H}$:

$$\pi(\tau_t(A)) = U_t^* \pi(A) U_t = e^{itH_\omega} \pi(A) e^{-itH_\omega}$$
$$H_\omega \geq 0, \quad H_\omega \Omega = 0$$

If there is only one ground state for τ_t, we necessarily have that it is a pure state (hence, π is irreducible) and that $\ker H_\omega = \mathbb{C}\Omega$.

Gapped ground states

Consider the case of a pure ground state with $\ker H_\omega = \mathbb{C}\Omega$. Then, for any $\gamma > 0$

$$\operatorname{spec} H_\omega \cap (0,\gamma) = \emptyset \text{ iff } \omega(A^*\delta(A)) \geq \gamma\omega(A^*A), A \in \mathcal{A}_{\mathrm{loc}} \text{ with } \omega(A) = 0$$

If this condition holds for some $\gamma > 0$, the ground state is gapped. Then

$$\operatorname{gap}(H_\omega) = \sup\{\gamma > 0 \mid \operatorname{spec} H_\omega \cap (0,\gamma) = \emptyset\}.$$

For infinite systems with Γ without boundary, e.g., $\Gamma = \mathbb{Z}^\nu$: $\operatorname{gap}(H_\omega)$ is the bulk gap. If Γ is a half-space of $\mathbb{Z}^\nu$, it may be referred to as the edge gap etc.

Stability of Spectral Gaps

Stability of the bulk gap

Suppose $\{h_x\}_{x\in\Gamma}$ defines generator δ with (for simplicity) a unique ground state ω and a gap $\gamma_0 > 0$:

$$\omega(A^*\delta(A)) \geq \gamma_0\omega(A^*A), A \in \operatorname{dom}\delta, \text{ with } \omega(A) = 0 \Leftrightarrow \operatorname{gap}(H_\omega) \geq \gamma_0.$$

Define perturbations of the form

$b \equiv B_x(n)$

$$h_x(s) = h_x + s\Phi_x, s \in \mathbb{R}, \Phi_x = \sum_n \Phi(b_x(n)), \text{ with } \|\Phi(b_x(n))\| \leq g(n).$$

The gap of the model is stable under such perturbations if for all $\gamma \in (0, \gamma_0)$, there exists $\boxed{s_0(\gamma) > 0}$ such that the gap for the perturbed model, γ_s, satisfies

$$\gamma_s \geq \gamma, \text{ for all } |s| < s_0(\gamma).$$

Stability theorem for frustration free finite range interactions

We consider perturbations of finite-range (R) frustration-free models with Hamiltonians of the form

$$H_\Lambda(s) = \sum_{x\in\Lambda} h_x + s \sum_{x\in\Lambda, n\geq 0} \Phi(b_x(n)) = \sum_{X\subset\Lambda} \Phi(s, X).$$

with uniformly bounded $h_x \in \mathcal{A}_{b_x(R)}$, $\sup_x \|h_x\| < \infty$. $\Gamma \subset \mathbb{R}^\nu$, Delone.
C1: There are $C > 0, q \geq 0$ such that $\operatorname{gap}(H_{b_x(n)}(0)) \geq Cn^{-q}$ (non-zero edge modes do not vanish faster than a power law).
C2: $\operatorname{gap}(H_{\omega_0}) = \gamma_0 > 0$.
C3: $\|\Phi(b_x(n))\| \leq \|\Phi\| e^{-an^\theta}$, for some $a > 0, \theta > 0$.
C4: LTQO. Denote by P_Λ the projection onto $\ker H_\Lambda(0)$. There exists a positive decreasing function G_0 for which, for all $A \in \mathcal{A}_{b_x(k)}$,

$$\|P_{b_x(m)} A P_{b_x(m)} - \omega_0(A) P_{b_x(m)}\| \leq \|A\|(k+1)^\nu G_0(m-k).$$

and

$$\sum_{n\geq 1} n^p G_0(n) < \infty, \text{ some } p > 4\nu + q$$

Counter example.

Spin $3/2$ chain ; $n=4$

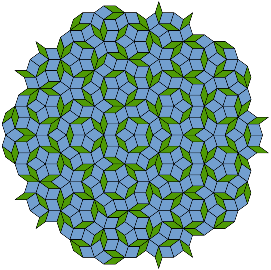
$$H = \sum_x P^{(3)}_{x,x+1}$$

g.s. highly (exponentially) degenerate

Knabe (1988) proved is gapped.
in the sense of finite volume Hamiltonians.

<u>but</u> gap is not stable

Not assuming a uniform gap in finite volume!



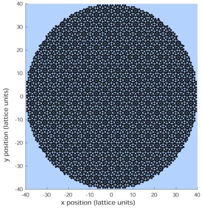


FIG. 2. Round section of the quasilattice, of radius 40 lattice units.

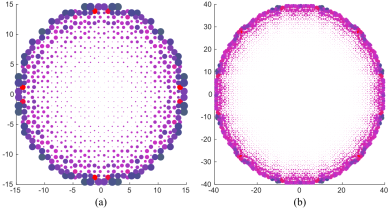


FIG. 7. (a) The radius is 15, with eigenvalue 0.018. (b) The radius is 40 with eigenvalue 0.007.

Figure: Penrose tiling. Ammann-Beenker tiling. Edges state or not? (T. Loring, J. Math. Phys. **60**, 081903 (2019))

h_x is frustration free $h_x \geq 0$

$$H_\Lambda = \sum_{\substack{x \in \Lambda \\ b_x(R) \subset \Lambda}} h_x \geq 0; \text{ FF} \Leftrightarrow \ker H_\Lambda \neq \{0\} \quad \forall \Lambda$$

Theorem

(Stability of the bulk gap, N-Sims-Young, arXiv:arXiv:2102.07209)

If conditions C1-C4 are satisfied, then, for all $\gamma \in (0, \gamma_0)$, there is a constant $\beta > 0$, such that the ground state, ω_s, for $\Phi(s)$, with

$$|s| \leq \frac{\gamma_0 - \gamma}{\beta \gamma_0}$$

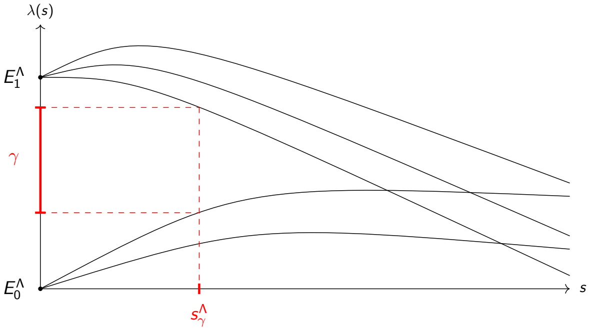
is unique, and the gap of $H_{\omega_s} > \gamma$.

Proved using the strategy of Bravyi-Hastings-Michalakis 2010, applied to the GNS Hamiltonian. β is explicit:

$$\beta = C_0 \sum_n n^{q+\nu} G_0(n).$$

For a model with a gap above the ground state to represent a gapped phase, the gap should be stable under a broad class of perturbations.

$$H_\Lambda(s) = H_\Lambda(0) + sV_\Lambda$$



The spectral gap of $H_\Lambda(s)$ above the 'ground state' is at least γ for all $0 \leq s \leq s_\gamma^\Lambda$.

Stability means that there is a Λ-independent lower bound for s_γ^Λ.

VI. Invariants. Ogata's construction for Symmetry Protected Topologic (SPT) Phases

SPT phases are gapped ground state phases defined by restricting to $\Phi \in \mathcal{B}_{a,\theta}^{\mathrm{gapped}}$ that share a symmetry given by a representation of a group G, and also requiting that the interpolating curves have that symmetry at every point.

Furthermore, one focusses on the trivial phase in $\mathcal{B}_{a,\theta}^{\mathrm{gapped}}$ (without symmetry condition).

Example

The AKLT chain (Affleck-Kennedy-Lieb-Tasaki 1987-88) is the spin-1 chain with nearest neighbor interaction given by

$$P^{AKLT}_{x,x+1} = \frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2$$

which is a 5-dim projection. In (Bachmann-N 2014) we constructed a C^1-curve of projections $P(s)$ such that $P(1) = P^{AKLT}$ and the model with nn interaction $P(0)$ has a unique product ground state in the TL and we show a uniform positive lower bound for the gap for $s \in [0,1]$. This implies that the AKLT chain belongs to the same phase as the model with a unique product ground state (the trivial phase).

In contrast, if we one restricts interpolations that respect spin rotation symmetry about 1 axis and an additional $\mathbb{Z}_2$ symmetry, an index argument shows that any curve connecting the AKLT model with a model in the trivial phase, must pass through a phase transition where the gap closes (Tasaki 2018, Ogata 2019-20). This implies that the AKLT chain belongs to a SPT phase distinct from the trivial phase.

The ‘Chen-Gu-Wen-Pollmann-Turner-Berg-Oshikawa-Ogata’ index

1. The AKLT chain

2. Ogata’s general construction

AKLT: $H_{[a,b]} = \sum_{x=a}^{b-1} P^{(2)}_{x,x+1}$ Spin-1

$\quad = \frac{1}{2} \vec{S}_x \cdot \vec{S}_{x+1} + \left(\frac{1}{6} \vec{S}_x \cdot \vec{S}_{x+1}\right)^2 + \frac{1}{3} \mathbb{1}$

has $SO(3)$ symmetry.

$\downarrow$

$U(g)$, $g \in SO(3)$; $\quad g = (\hat{n}, \theta)$

$\hat{n} \in \mathbb{R}^3$, $\|\hat{n}\| = 1$; $\quad \theta \in [0, 2\pi)$

$$[P^{(2)}, U(g) \otimes U(g)] = 0$$

dim ker $H_{[a,b]} = 4$

$\alpha, \beta \in \{\pm\}$

$$\Psi^{\alpha,\beta}_{[1,n]} = \overbrace{\langle\alpha| \otimes id \otimes |\beta\rangle \left(\underset{=}{\mathbb{I}_2} \otimes P^{(1)} \otimes \cdots \otimes P^{(1)} \otimes \underset{=}{\mathbb{I}_2}\right) \underbrace{\varphi \otimes \cdots \otimes \varphi}_{n+1}}^{\in (\mathbb{C}^3)^{\otimes n}}$$

$\in \ker H_{[1,n]}$

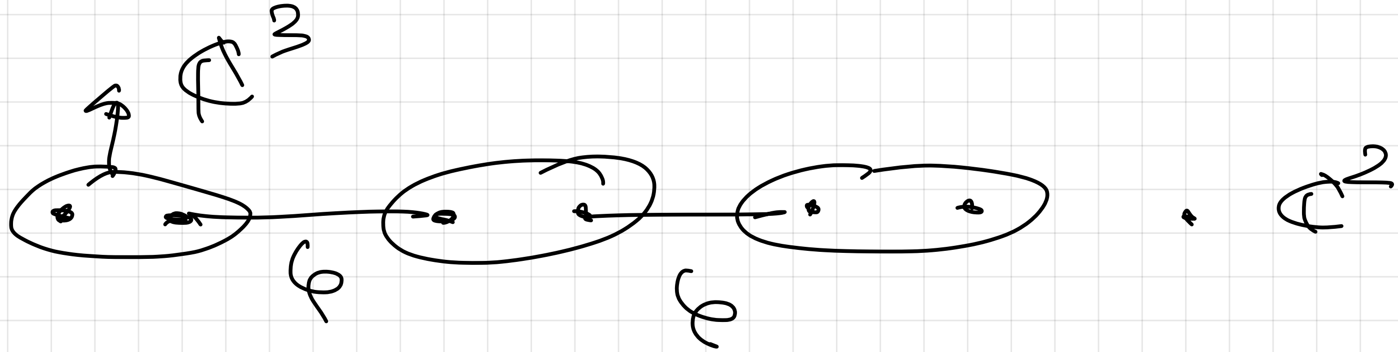
$\in (\mathbb{C}^2)^{\otimes 2(n+1)}$

$\varphi = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$

$P^{(1)} : \underset{=}{\mathbb{C}^2} \otimes \underset{=}{\mathbb{C}^2} \longrightarrow \mathbb{C}^3$: onto spin 1

spin $1/2 \otimes$ spin $1/2$ = spin $0 \oplus$ spin 1

<u>VBS</u>



$v(g)$: spin $1/2$ rep (of $SU(2)$)

$$\underline{P^{(1)}\left(v(g)\otimes v(g)\right) = U(g)\, P^{(1)}}$$

$$g = e^{i\theta\, \hat{n}\cdot\bar{S}}$$

<u>Ex</u> $$\left(U(g)\right)^{\otimes n} \Psi_{[1,n]}^{|\alpha\rangle,|\beta\rangle} = \Psi_{[1,n]}^{v^*(g)|\alpha\rangle,\, v(g)|\beta\rangle}$$

Symmetry acts on Bex $\mathcal{H}_{[1,n)}$ as $\underline{v^*(g)\otimes v(g)}$

" $$D^{(1/2)}\left(e^{i\, 2\pi\, \hat{n}\cdot\bar{S}}\right) = -\mathbb{I}_2$$ "

to "isolate" one factor of $\nu(g)$

take lim $\psi^{\alpha,\beta}_{[1,n]} \to \psi^{\alpha}_{[1,\infty)}$
$n \to \infty$

Projective representations of a group G

$$U_g U_h = \underline{c(g,h)}\, U_{gh} \qquad c(g,h) \in U(1)$$

$$U_g U_{h_1} U_{h_2} = c(h_1,h_2)\, U_g\, U_{h_1 h_2} = \underline{c(h_1,h_2)\, c(g,h_1 h_2)}\, U_{g h_1 h_2}$$

$$\|$$

$$c(g,h_1)\, U_{gh_1} U_{h_2} = \underline{\underline{c(g,h_1)\, c(gh_1,h_2)}}\, U_{g h_1 h_2}$$

$\to$ 2-cocycle. $\tilde{c}(g,h) = \varphi(g)\varphi(h)c(g,h)$

equivalence class for a group $\underline{H^2(G, U(1))}$

has a abelian group structure.

$\mathbb{Z}_2 \times \mathbb{Z}_2 \subset SU(2)$

$$H^2(\underline{\mathbb{Z}_2 \times \mathbb{Z}_2}, U(1)) = \mathbb{Z}_2$$

$\{0, 1\}$

VI.2 Ogata's construction of an SPT invariant for quantum spin chains

Setting: $\Gamma = \mathbb{Z}$, an interaction Φ with a unique gapped ground state in the trivial phase.

Concretely, $\Phi \in \mathcal{B}_{a,\theta}^{\mathrm{gapped}}$, and Φ is connected by a differentiable gapped path to Φ_0, defined by

$$\Phi_0(X) = 0, \text{ unless } X = \{x\}, x \in \Gamma, \text{ and } \Phi_0(\{x\}) = (\mathbb{1} - |0\rangle\langle 0|)_x.$$

Φ_0 has a unique gapped ground state given by the product state

$$\bigotimes_x \langle 0| \cdot |0\rangle.$$

Φ is assumed to have a local symmetry given by unitary representations $U_x(g)$ of a group G. For the infinite chain this symmetry is described by the automorphisms

$$\beta_g(A) = \left(\bigotimes_x U_x(g)^*\right) A \left(\bigotimes_x U_x(g)\right), A \in \mathcal{A}_{\mathbb{Z}}.$$

Φ has the G-symmetry $(\beta_g(\Phi(X)) = \Phi(X), \forall X.$

For Symmetry Protected Phases, we define equivalence by only using differentiable paths of interactions that all have the same G-symmetry.

We want an invariant for the resulting equivalence classes of G-symmetric interactions with a unique gapped ground state which is equivalent to Φ_0 (without the symmetry).

Theorem (Ogata 2020)

There exists an $H^2(G, U(1))$-valued invariant for the SPT equivalence classes.

Coincides with the invariants given by Pollmann-Turner-Berg-Oshikawa, 2010-11 and Chen-Gu-Wen, 2020-11 in a more restricted context.

The $H^2(G, U(1))$-valued invariant classifies the projective representations of G. The proof of the theorem is by constructing such a representation.

Inspired by what we found for the AKLT chain, we look for a unitary implementation of the symmetry G on a half-chain.

Starting point: $\Phi \sim \Phi_0$ implies the existence of a gapped interpolating path $\Phi(s), s \in [0,1]$, $\Phi(0) = \Phi_0, \Phi(1) = \Phi$, and an associated interaction $\Psi(s)$, that generates the quasi-adiabatic evolution α_s. In particular

product state

$$\underline{\underline{\omega_1}} = \underline{\omega_0 \circ \alpha_1}.$$

Important property: $\Psi \in \mathcal{B}_{a',\theta'}([0,1])$, i.e., of fast decay. This means $\Psi(s)$ can be decoupled by a bounded perturbation.

$\Gamma = \mathbb{Z} = \Gamma_L \sqcup \Gamma_R$

Define $\Gamma_L = \underline{(-\infty, 0]}$ and $\Gamma_R = [1, \infty)$ and $\underline{\tilde{\Psi}}(s)$ such that

$$\Psi(s) = \tilde{\Psi}(s) + \underline{V(s)}, \text{ such that } \underline{\tilde{\Psi}(s, X)} \neq 0 \implies X \subset \Gamma_L \text{ or } X \subset \Gamma_R.$$

Considering $V(s)$ as a perturbation and using interaction picture gives unitaries $U(s)$ s.t.

generated by $\tilde{\Psi}(s)$

$\Psi(s)$

$$\alpha_s = \underline{\underline{\tilde{\alpha}_s}} \circ \underline{\mathrm{Ad}\, U(s)}$$

$U(s)$ is the solution of

$\mathrm{Ad}\, U(A) = U^* A U$

$$\frac{d}{ds} = -iV^{\mathrm{int}}(s)U(s), \quad U(0) = \mathbb{1}$$

and $\underline{V^{\mathrm{int}} = \tilde{\alpha}_s^{-1}(V(s))}$.

Since ω_0 is product and $\tilde{\alpha}_s = \tilde{\alpha}_s^L \otimes \tilde{\alpha}_s^R$, we have

$\omega_1^L = \omega_0|_{\mathcal{Q}_L} \circ \tilde{\alpha}_1^L$

$$\omega_1 = \omega_0 \circ \alpha_1 = \omega_0 \circ \tilde{\alpha}_1 \circ \mathrm{Ad}\, U_1.$$

↳ product

In other words, we have states ω_1^L and ω_1^R of the half-chains such that

(Split property)

$$\omega_1 = \omega_1^L \otimes \omega_1^R \circ \mathrm{Ad}\, U_1 \tag{1}$$

In particular $\omega_1 \underset{\mathrm{u}}{\sim} \omega_1^L \otimes \omega_1^R$ and also $\omega_1 \circ \beta_g \underset{\mathrm{u}}{\sim} (\omega_1^L \otimes \omega_1^R) \circ \beta_g$.

Next, recall $\omega_1 \circ \beta_g = \omega_1$ and $\beta_g = \beta_g^L \otimes \beta_g^R$.

Therefore,

$\underset{\mathrm{u}}{\sim}\ \omega_1 \circ \beta_g \ \overset{=}{\underset{\mathrm{u}}{\sim}}$

$$(\omega_1^L \otimes \omega_1^R) \circ \beta_g = (\omega_1^L \circ \beta_g^L) \otimes (\omega_1^R \circ \beta_g^R) \underset{\mathrm{u}}{\sim} \omega_1 \underset{\mathrm{u}}{\sim} \omega_1^L \otimes \omega_1^R$$

This gives

$$\omega_1^R \circ \beta_g^R \underset{\mathrm{u}}{\sim} \omega_1^R$$

and also

$$\omega_1 \circ \beta_g^R \ \underset{\mathrm{u}}{\sim} \ \omega_1^L \otimes (\omega_1^R \circ \beta_g^R) \underset{\mathrm{u}}{\sim} \omega_1^L \otimes \omega_1^R \underset{\mathrm{u}}{\sim} \omega_1$$

The lefthand side is unitarily equivalent to $\omega_1 \circ \beta_g^R$, due to (1).

$$\omega_1 \underset{u}{\sim} \omega_1^L \otimes \omega_1^R$$

$$\pi \to \pi \circ \beta_g$$

$$\omega_1(A) = \langle \Omega, \pi(A)\Omega \rangle$$

$$\omega_1^L \otimes \omega_1^R = \langle \Omega, \pi(U)\,\pi(A)\,\pi(U^*)\,\Omega \rangle$$

$$= \langle \tilde{\Omega}, \pi(A)\,\tilde{\Omega} \rangle$$

Conclusion:

$$\omega_1 \circ \beta_g^R \underset{\mathrm{u}}{\sim} \omega_1$$

This means that β_g^R is implemented by a unitary $\boxed{U_g^R}$ in the GNS representation of ω_1:

$$\omega_1 \circ \beta_g^R(A) = \langle \Omega, (U_g^R)^* \pi(A) U_g^R \Omega \rangle .$$

$\beta_g \circ \beta_h = \beta_{gh}$
$\forall g, h \in G$

Since π is an irreducible representation and

$$(U_h^R)^*(U_g^R)^* \pi(\cdot) U_g^R U_h^R = \boxed{\pi \circ \beta_{gh}} = (U_{gh}^R)^* \pi(\cdot) U_{gh}^R$$

whence

$$U_{gh}^R (U_h^R)^* (U_g^R)^* \pi(\cdot) U_g^R U_h^R (U_{gh}^R)^* = \pi(\cdot),$$

we must gave $c(g,h) \in U(1)$ s.t.

$$U_g^R U_h^R (U_{gh}^R)^* = c(g,h) \mathbb{1}$$

with c belonging to an equivalence class of 2-cycles labeled by an element of $H^2(G, U(1))$.

$H^2(SO(3), U(1)) = \mathbb{Z}_2$

generalizations: Ogata

- 2 dim quantum spin systems

$$H^3(G, U(1))$$

- 1 dim fermion chains

- - -

- 2 dim fermion systems

. - .

Recent results on the phase diagram of $O(n)$ spin chains

$O(n)$ chains: $\Gamma = \mathbb{Z}$, $\mathcal{H}_x = \mathbb{C}^n$.

AKLT model, $n = 3$: only non-zero interactions are

$\Phi(\{x, x+1\}) = h_{x,x+1} = \frac{1}{2}\boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1} + \frac{1}{6}(\boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1})^2 + \frac{1}{3}\mathbb{1} = P^{(2)}_{x,x+1}$.

The general isotropic nearest neighbor interaction for $n = 3$:

$h_{x,x+1} = \cos\phi \boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1} + \sin\phi(\boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1})^2$.

Alternative way to represent the AKLT Hamiltonian in terms of 'swap' operator, T, and a rank-1 projection:

$T(\varphi \otimes \psi) = \psi \otimes \varphi$

$$2P^{(2)} = T - 2P^{(0)} + \mathbb{1},$$

where $P^{(0)}$ projects onto the singlet state. There is an o.n. basis e_1, e_0, e_{-1} such that

$$\psi = \frac{1}{\sqrt{3}}(e_1 \otimes e_1 + e_0 \otimes e_0 + e_{-1} \otimes e_{-1}).$$

This generalizes to n-dimensional spins and arbitrary coupling constants as follows

$$uT + vQ, \quad u, v \in \mathbb{R}$$

where Q is the projection to

$$\psi = \frac{1}{\sqrt{n}} \sum_{\alpha=1}^{n} |\alpha, \alpha\rangle.$$

Phase diagrams. The stability of gapped ground states

The gapped ground state phases are open regions in space of interactions, not isolated special points, meaning they are stable.
For gapped, frustration-free models satisfying no-local order condition good general stability results exists:

Yarotsky 2006, Bravyi-Hastings-Michalakis 2010, Michalakis-Zwolak 2013, Szehr-Wolf 2015, Fröhlich-Pizzo (et al.) 2018-20, N-Sims-Young 2021.

These results prove the AKLT point is part of an open region on the red phase of the $n = 3$ phase diagram.
The uniqueness condition of the gapped ground state can be relaxed (N-Sims-Young 2021) but we have no general stability results yet that do not require frustration free property.

The point $u = 0, v = -1$, where we have dimerization, is not frustration free:

$$\langle h_{x,x+1} \rangle > \inf \mathrm{spec}(h_{x,x+1}).$$

Recent proof of open gapped region in phase diagram for large n (Björnberg-Mühlbacher-N-Ueltschi, 2021).

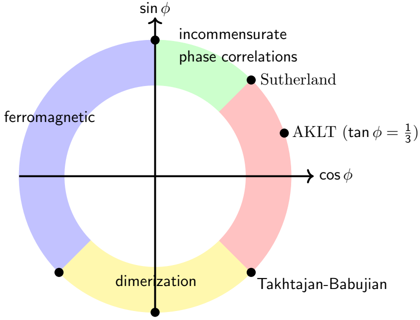


Figure: Ground state phase diagram for the $S = 1$ chain ($n = 3$) with nearest-neighbor interactions $\cos\phi \boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1} + \sin\phi(\boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1})^2$.

- $\phi = 0$ Heisenberg AF chain, Haldane phase (Haldane, 1983)
- $\tan\phi = 1/3$, AKLT point (Affleck-Kennedy-Lieb-Tasaki, 1987,1988), FF, MPS, gapped
- $\tan\phi = 1$, solvable, gapless, SU(3) invariant, (Sutherland, 1975)
- $\phi \in [\pi/2, 3\pi/2]$, ferromagnetic, FF, gapless
- $\phi = -\pi/2$, solvable, SU(3) invariant, Temperley-Lieb algebra, dimerized, gapped (Klümper; Affleck,1990)
- $\phi - = -\pi/4$ gapless, Bethe-ansatz, (Takhtajan; Babujian, 1982)

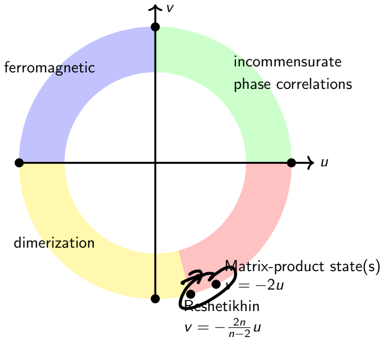


Figure: Ground state phase diagram for the chain with nearest-neighbor interactions $uT + vQ$ for $n \geq 3$, studied by Tu & Zhang, 2008.

- $v = -2nu/(n-2), n \geq 3$, Bethe ansatz point (Reshetikhin, 1983)
- $v = -2u$: frustration free point, equivalent to $\perp$ projection onto symmetric vectors $\ominus$ one. Unique g.s. if n odd; two 2-periodic g.s. for even n; spectral gap in all cases and stable phase (N-Sims-Young, 2021).
- $u = 0, v = -1$. Equivalent to the $SU(n) - P^{(0)}$ models aka Temperley-Lieb chain; Affleck, 1990, Nepomechie-Pimenta 2016). Dimerized for all $n \geq 3$ (Aizenman, Duminil-Copin, Warzel, 2020); 'Stability' for large n (Björnberg-Mühlbacher-N-Ueltschi, 2021).

$n=2$ $\quad \pm \sum_x \vec{S}_x \cdot \vec{S}_x$